

# THE DENOMINATORS OF CONVERGENTS FOR CONTINUED FRACTIONS

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ABSTRACT. For any real number  $x \in [0, 1)$ , we denote by  $q_n(x)$  the denominator of the  $n$ -th convergent of the continued fraction expansion of  $x$  ( $n \in \mathbb{N}$ ). It is well-known that the Lebesgue measure of the set of points  $x \in [0, 1)$  for which  $\log q_n(x)/n$  deviates away from  $\pi^2/(12 \log 2)$  decays to zero as  $n$  tends to infinity. In this paper, we study the rate of this decay by giving an upper bound and a lower bound. What is interesting is that the upper bound is closely related to the Hausdorff dimensions of the level sets for  $\log q_n(x)/n$ . As a consequence, we obtain a large deviation type result for  $\log q_n(x)/n$ , which indicates that the rate of this decay is exponential.

## 1. INTRODUCTION

Let  $T : [0, 1) \rightarrow [0, 1)$  be the *continued fraction transformation* defined as

$$T(0) := 0 \quad \text{and} \quad T(x) := 1/x - [1/x] \quad \text{if } x \in (0, 1).$$

where  $[x]$  denotes the greatest integer not exceeding  $x$ . Then every real number  $x \in [0, 1)$  can be uniquely written as

$$x = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \cdots + \frac{1}{a_n(x) + \cdots}}}, \quad (1.1)$$

where  $a_1(x) = [1/x]$  and  $a_{n+1}(x) = a_1(T^n x)$  for all  $n \geq 1$ . The representation (1.1) is said to be the *continued fraction expansion* of  $x$  and  $a_n(x), n \geq 1$  are called the *partial quotients* of the continued fraction expansion of  $x$ . Sometimes we write the form (1.1) as  $[a_1(x), a_2(x), \dots, a_n(x), \dots]$ . For any  $n \geq 1$ , we denote by  $\frac{p_n(x)}{q_n(x)} := [a_1(x), a_2(x), \dots, a_n(x)]$  the  $n$ -th *convergent* of the continued fraction expansion of  $x$ , where  $p_n(x)$  and  $q_n(x)$  are relatively prime. With the conventions  $p_{-1} = 1, q_{-1} = 0, p_0 = 0, q_0 = 1$ , the quantities  $p_n$  and  $q_n$  satisfy the following recursive formula:

$$p_n(x) = a_n(x)p_{n-1}(x) + p_{n-2}(x) \quad \text{and} \quad q_n(x) = a_n(x)q_{n-1}(x) + q_{n-2}(x). \quad (1.2)$$

It is easy to see that these convergents are rational numbers and  $p_n(x)/q_n(x) \rightarrow x$  as  $n \rightarrow \infty$  for all  $x \in [0, 1)$ . More precisely,

$$\frac{1}{2q_{n+1}^2(x)} \leq \frac{1}{2q_n(x)q_{n+1}(x)} \leq \left| x - \frac{p_n(x)}{q_n(x)} \right| \leq \frac{1}{q_n(x)q_{n+1}(x)} \leq \frac{1}{q_n^2(x)}. \quad (1.3)$$

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This is to say that the speed of  $p_n(x)/q_n(x)$  approximating to  $x$  is dominated by  $q_n^{-2}(x)$ . So the denominator of the  $n$ -th convergent  $q_n(x)$  plays an important role in the problem of Diophantine approximation. For more details about continued fractions, we refer the reader to two monographs of Iosifescu and Kraaikamp [11] and Khintchine [13].

For an irrational number  $x \in [0, 1)$ , we denote

$$\beta_*(x) = \liminf_{n \rightarrow \infty} \frac{\log q_n(x)}{n} \quad \text{and} \quad \beta^*(x) = \limsup_{n \rightarrow \infty} \frac{\log q_n(x)}{n}.$$

The functions  $\beta_*(x)$  and  $\beta^*(x)$  are called the *lower* and *upper Lévy constant* of  $x$  respectively. If  $\beta_*(x) = \beta^*(x)$ , we say that  $x$  has a Lévy constant and denote the common value by  $\beta(x)$ . It is not difficult to check that  $\beta_*(x) \geq \gamma_0 := \log((\sqrt{5}+1)/2)$  for all irrational number  $x$ . On the one hand, Faivre [5] showed that every quadratic irrational has a Lévy constant. In 2006, Wu [20] proved that the set of Lévy constants of quadratic irrationals is dense in the interval  $[\gamma_0, +\infty)$ . On the other hand, Faivre [6] showed that for any  $\gamma \geq \gamma_0$ , there exists an irrational number  $x$  such that  $x$  has Lévy constant  $\lambda$ . Recently, Baxa [3] improved this result for transcendental numbers. That is to say, there exists a transcendental number  $x$  such that  $\beta(x) = \gamma$  for any  $\gamma \geq \gamma_0$ . Also, Baxa [2] obtained that for any two real numbers satisfying  $\gamma_0 \leq \gamma_1 \leq \gamma_2 < +\infty$ , there exist non-denumerably many pairwise not equivalent irrational numbers  $x$  such that  $\beta_*(x) = \gamma_1$  and  $\beta^*(x) = \gamma_2$ . Furthermore, Wu [19] considered the Hausdorff dimension of the set of such points and gave it a lower bound. A basic result about Lévy constant is due to Lévy [14], who proved that the function  $\beta(x)$  is constantly  $\pi^2/(12 \log 2)$  for  $\lambda$ -almost all  $x \in [0, 1)$ . Here  $\lambda$  denotes the Lebesgue measure on  $[0, 1)$ .

**Theorem 1.1** ([14]). *For  $\lambda$ -almost all  $x \in [0, 1)$ ,*

$$\lim_{n \rightarrow \infty} \frac{\log q_n(x)}{n} = \frac{\pi^2}{12 \log 2}.$$

From the fractal dimension points of view, Barreira and Schmeling [1] pointed out that the set of points  $x \in [0, 1)$  for which the limit in Theorem 1.1 does not exist (i.e.,  $\beta_*(x) < \beta^*(x)$ ) has full Hausdorff dimension. Furthermore, Pollicott and Weiss [18] first considered the multifractal analysis of  $\beta(x)$  and proved that the spectral function

$$\tau(\gamma) := \dim_{\text{H}} \{x \in [0, 1) : \beta(x) = \gamma\} = \frac{\inf_{\theta \in \mathbb{R}} \{\theta \cdot 2\gamma + P(\theta)\}}{2\gamma}$$

for any  $\gamma \geq \gamma_0$  (see also Fan et al. [7] and Kesseböhmer and Stratmann [12]), where  $\dim_{\text{H}}$  denotes the Hausdorff dimension and  $P(\cdot)$  is called the *Diophantine pressure function* given by

$$P(\theta) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{a_1, \dots, a_n} q_n^{-2\theta}([a_1, \dots, a_n]) \quad \text{for any } \theta > 1/2.$$

It is worth remarking that the spectral function  $\tau(\cdot)$  is real-analytic on the interval  $(\gamma_0, +\infty)$  satisfying  $\tau(\gamma)$  goes to  $1/2$  as  $\gamma$  tends to infinity, it is increasing on the interval  $[\gamma_0, \pi^2/(12 \log 2)]$  and decreasing on the interval  $[\pi^2/(12 \log 2), +\infty)$ , and it also has a unique maximum value equal to 1 at point  $\pi^2/(12 \log 2)$ ; the Diophantine

pressure function  $P(\cdot)$  has a singularity at  $1/2$  and is decreasing, convex and real-analytic on  $(1/2, +\infty)$  satisfying

$$P(1) = 0 \quad \text{and} \quad P'(1) = -\pi^2/(6 \log 2). \quad (1.4)$$

More detailed analyses of  $\tau(\cdot)$  and  $P(\cdot)$  can be founded in Fan et al. [7], Kesseböhmer and Stratmann [12], Mayer [15] and Pollicott and Weiss [18]. From the metrical points of view, some limit theorems about  $q_n(x)$  have been extensively investigated. For instance, Ibragimov [10] proved that the distribution of the  $\log q_n(x)$ , suitably normalized, converges to the normal distribution with mean 0 and unit variance, that is, the central limit theorem for  $q_n(x)$ . Furthermore, Morita [16] showed that the Berry-Esseen bound for the above central limit theorem is as we would expect  $\mathcal{O}(n^{-1/2})$ . Later, Philipp and Stackelberg [17] obtained the classical law of the iterated logarithm for  $q_n(x)$  (see also Gordin and Reznik [9]).

It is worth noting that these classical limit theorems basically concern that the averages taken over large samples converge to expectation values in some sense, but say little or nothing about the rate of convergence. It follows from Theorem 1.1 that the Lebesgue measure of the set of points  $x$  for which  $\log q_n(x)/n$  deviates away from  $\pi^2/(12 \log 2)$  decays to zero as  $n$  goes to infinity. A natural question is arisen: what are the rates of these decreasing probabilities? In fact, Fang et al. [8] have considered these decays and showed that the upper bounds of these decays are exponential. In this paper, we not only obtain the upper and lower bounds of these decreasing probabilities, but also give them explicit formulae. And an interesting phenomenon is that the explicit formula of the upper bound is closely related to the spectral function  $\tau(\cdot)$  (see Remarks 2.2 and 2.4 below).

## 2. MAIN RESULTS

In this section, we will state our main results. For simplicity, we use the notation  $b$  to denote the constant  $\pi^2/(12 \log 2)$ .

**Theorem 2.1.** *For any  $\varepsilon > 0$ , we have*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in [0, 1) : \frac{\log q_n(x)}{n} \geq b + \varepsilon \right\} \leq \theta_1(\varepsilon)$$

and

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in [0, 1) : \frac{\log q_n(x)}{n} \geq b + \varepsilon \right\} \geq -2 \log b_\varepsilon - \log 3$$

where  $\theta_1(\varepsilon) = \inf_{0 < t < 1} \{ -t(b + \varepsilon) + P(1 - t/2) \} < 0$  and  $b_\varepsilon$  denotes the smallest integer no less than  $e^{b+\varepsilon}$ .

**Remark 2.2.** By the domain of the function  $P(\cdot)$ , we can write  $\theta_1(\varepsilon)$  as

$$\theta_1(\varepsilon) = \inf_{t < 1} \{ -t(b + \varepsilon) + P(1 - t/2) \}.$$

In fact, for any  $\varepsilon > 0$ , let  $f(t) = -t(b + \varepsilon) + P(1 - t/2)$  for any  $t \leq 0$ . Since  $P(\cdot)$  is convex and real-analytic on  $(1/2, +\infty)$ , we know  $P'(1 - t/2) \geq P'(1)$  for any  $t \leq 0$ . It follows from (1.4) that

$$f'(t) = -(b + \varepsilon) - 2^{-1} \cdot P'(1 - t/2) \leq -(b + \varepsilon) - 2^{-1} \cdot P'(1) = -\varepsilon < 0$$

for any  $t \leq 0$ . So  $f(\cdot)$  is non-increasing on  $(-\infty, 0]$  and hence  $f(t) \geq 0$  for any  $t \leq 0$ . As a consequence, it is easy to check that

$$\theta_1(\varepsilon) = 2(b + \varepsilon)(\tau(b + \varepsilon) - 1)$$

and hence that  $-(b + \varepsilon) < \theta_1(\varepsilon) < 0$  and  $\theta_1(\varepsilon)$  tends to zero as  $\varepsilon$  goes to zero since the spectral function  $\tau(\cdot)$  has a unique maximum value equal to 1 at point  $b$ .

**Theorem 2.3.** *For any  $0 < \varepsilon \leq b$ , we have*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in [0, 1) : \frac{\log q_n(x)}{n} \leq b - \varepsilon \right\} \leq \theta_2(\varepsilon)$$

and for any  $0 < \varepsilon \leq b - \log 2$ ,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in [0, 1) : \frac{\log q_n(x)}{n} \leq b - \varepsilon \right\} \geq -2 \log b_\varepsilon^* - \log 3,$$

where  $\theta_2(\varepsilon) = \inf_{t > 0} \{t(b - \varepsilon) + P(1 + t/2)\} < 0$  and  $b_\varepsilon^*$  denotes the largest integer no greater than  $(e^{b-\varepsilon} - 1)$ .

**Remark 2.4.** Being similar to Remark 2.2,  $\theta_2(\varepsilon)$  can be written as

$$\theta_2(\varepsilon) = \inf_{t > -1} \{t(b - \varepsilon) + P(1 + t/2)\}$$

for any  $0 < \varepsilon \leq b$ . Moreover, it also has another alternative form

$$\theta_2(\varepsilon) = 2(b - \varepsilon)(\tau(b - \varepsilon) - 1),$$

which only holds for  $0 < \varepsilon \leq b - \log((\sqrt{5} + 1)/2)$  from the definition of  $\tau(\cdot)$ . In this case, it is clear to see that  $-2(b - \varepsilon) \leq \theta_2(\varepsilon) < 0$  and  $\theta_2(\varepsilon)$  tends to zero as  $\varepsilon$  goes to zero.

The following is a result of large deviations for  $\log q_n(x)/n$ , which improves the result of Theorem 1.1 by Borel-Cantelli lemma.

**Theorem 2.5.** *For any  $\varepsilon > 0$ , there exist constants  $A, B > 0$  and  $\alpha, \beta > 0$  (both only depending on  $\varepsilon$ ) such that for all  $n \geq 1$ , we have*

$$Be^{-\beta n} \leq \lambda \left\{ x \in [0, 1) : \left| \frac{\log q_n(x)}{n} - \frac{\pi^2}{12 \log 2} \right| \geq \varepsilon \right\} \leq Ae^{-\alpha n}.$$

### 3. THE PROOFS OF THEOREMS

This section is devoted to giving the proofs of our main results. We denote by  $\mathbb{I}$  the set of all irrational numbers in  $[0, 1)$  and use the notation  $E(\xi)$  to denote the expectation of a random variable  $\xi$  w.r.t. the Lebesgue measure  $\lambda$ . For any  $n \in \mathbb{N}$  and  $a_1, a_2, \dots, a_n \in \mathbb{N}$ , we call

$$I(a_1, \dots, a_n) := \{x \in \mathbb{I} : a_1(x) = a_1, \dots, a_n(x) = a_n\}$$

the  $n$ -th order *cylinder* of continued fractions. It is well-known (see [4, 11]) that  $I(a_1, \dots, a_n)$  is an interval with the endpoints  $p_n q_n^{-1}$  and  $(p_n + p_{n-1})(q_n + q_{n-1})^{-1}$ . As a consequence, the length of  $I(a_1, \dots, a_n)$  denoted by  $|I(a_1, \dots, a_n)|$ , is equal to  $q_n^{-1}(q_n + q_{n-1})^{-1}$ , where the quantities  $p_n$  and  $q_n$  are obtained by the recursive formula (1.2). The following lemma establishes a relation between the Diophantine pressure function  $P(\cdot)$  and the growth of the expectation of  $q_n$ , which plays an important role in our proofs.

**Lemma 3.1.** *For any  $\theta < 1/2$ ,*

$$P(1 - \theta) = \lim_{n \rightarrow \infty} \frac{1}{n} \log E(q_n^{2\theta}).$$

*Proof.* By the definition of expectation, we know that

$$E(q_n^{2\theta}) = \sum_{a_1, \dots, a_n} q_n^{2\theta}([a_1, \dots, a_n]) \cdot \lambda(I(a_1, \dots, a_n)), \quad (3.1)$$

where  $a_1, \dots, a_n$  run over all the positive integers. Since

$$\frac{1}{2q_n^2([a_1, \dots, a_n])} \leq \lambda(I(a_1, \dots, a_n)) = |I(a_1, \dots, a_n)| \leq \frac{1}{q_n^2([a_1, \dots, a_n])},$$

combing this with (3.1), we deduce that

$$\frac{1}{2} \cdot \sum_{a_1, \dots, a_n} q_n^{-2(1-\theta)}([a_1, \dots, a_n]) \leq E(q_n^\theta) \leq \sum_{a_1, \dots, a_n} q_n^{-2(1-\theta)}([a_1, \dots, a_n])$$

and hence that

$$P(1-\theta) = \lim_{n \rightarrow \infty} \frac{1}{n} \log E(q_n^{2\theta}) \quad \text{for any } \theta < 1/2.$$

□

**3.1. Proof of Theorem 2.1.** The proof is divided into two parts: limsup part and liminf part. The proof of limsup part heavily relies on the Markov's inequality and Lemma 3.1. The idea of the proof of liminf part is from finding a subset inside whose Lebesgue measure decays to 0 exponentially.

*Proof of the limsup part.* Let  $0 < t < 1$  be a parameter. Notice that

$$\lambda \left\{ x \in \mathbb{I} : \frac{\log q_n(x)}{n} \geq b + \varepsilon \right\} = \lambda \left\{ x \in \mathbb{I} : q_n^t(x) \geq e^{nt(b+\varepsilon)} \right\},$$

in view of Markov's inequality, we have that

$$\lambda \left\{ x \in \mathbb{I} : \frac{\log q_n(x)}{n} \geq b + \varepsilon \right\} \leq e^{-nt(b+\varepsilon)} \cdot E(q_n^t). \quad (3.2)$$

By Lemma 3.1, we know

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log E(q_n^t) = P(1-t/2).$$

Hence, for any  $\eta > 0$ , there exists a positive number  $N$  (depending on  $\eta$ ) such that for all  $n \geq N$ , we have

$$E(q_n^t) \leq e^{n(P(1-t/2)+\eta)}.$$

Fixed such  $n \geq N$ , it follows from (3.2) that

$$\lambda \left\{ x \in \mathbb{I} : \frac{\log q_n(x)}{n} \geq b + \varepsilon \right\} \leq e^{-nt(b+\varepsilon)+n(P(1-t/2)+\eta)}. \quad (3.3)$$

Taking the logarithm on both sides of the inequality (3.3), we deduce that

$$\frac{1}{n} \log \lambda \left\{ x \in \mathbb{I} : \frac{\log q_n(x)}{n} \geq b + \varepsilon \right\} \leq -t(b+\varepsilon) + P(1-t/2) + \eta.$$

Thus, for all  $0 < t < 1$ , we obtain that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in \mathbb{I} : \frac{\log q_n(x)}{n} \geq b + \varepsilon \right\} \leq -t(b+\varepsilon) + P(1-t/2)$$

since  $\eta > 0$  is arbitrary. Therefore,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in \mathbb{I} : \frac{\log q_n(x)}{n} \geq b + \varepsilon \right\} \leq \theta_1(\varepsilon)$$

with

$$\theta_1(\varepsilon) = \inf_{0 < t < 1} \{ -t(b + \varepsilon) + P(1 - t/2) \}.$$

Now it remains to show that  $\theta_1(\varepsilon) < 0$ . In fact, let  $h(u)$  be the function defined as

$$h(u) = -u(b + \varepsilon) + P(1 - u/2) \text{ for any } u < 1.$$

In view of (1.4), it is easy to check that  $h(0) = 0$  and  $h'(0) = -\varepsilon < 0$ . Hence there exists  $u_0 > 0$  such that  $h(u_0) < 0$  by the definition of derivative. Therefore, we complete the proof of the limsup part by observing that  $\theta_1(\varepsilon) \leq h(u_0) < 0$ .  $\square$

To prove the liminf part, we need the following lemma (see [13]).

**Lemma 3.2** ([13]). *For any  $n \geq 1$  and  $a_1, \dots, a_n, a_{n+1} \in \mathbb{N}$ , we have*

$$\frac{1}{3a_{n+1}^2} \leq \frac{|I(a_1, \dots, a_n, a_{n+1})|}{|I(a_1, \dots, a_n)|} \leq \frac{2}{a_{n+1}^2}.$$

*Proof.* For any  $n \geq 1$  and  $a_1, \dots, a_n, a_{n+1} \in \mathbb{N}$ , we know that

$$|I(a_1, \dots, a_n)| = \frac{1}{q_n(q_n + q_{n-1})} \text{ and } |I(a_1, \dots, a_n, a_{n+1})| = \frac{1}{q_{n+1}(q_{n+1} + q_n)},$$

where the quantities  $p_{n-1}$ ,  $q_{n-1}$ ,  $p_n$ ,  $q_n$ ,  $p_{n+1}$  and  $q_{n+1}$  satisfy the recursive formula (1.2). Therefore,

$$\frac{|I(a_1, \dots, a_n, a_{n+1})|}{|I(a_1, \dots, a_n)|} = \frac{q_n(q_n + q_{n-1})}{q_{n+1}(q_{n+1} + q_n)} = \frac{1}{a_{n+1}^2} \cdot \frac{1 + y_n}{(1 + z_n)(1 + 1/a_{n+1} + z_n)}, \quad (3.4)$$

where  $y_n = q_{n-1}/q_n \in [0, 1)$ ,  $z_n = y_n/a_{n+1}$  and the last equation follows from the recursive formula  $q_{n+1} = a_{n+1}q_n + q_{n-1}$ . The second factor on the last term of (3.4) is obviously not greater than 2 since  $y_n < 1$ ,  $a_{n+1} > 0$  and  $z_n > 0$ . Notice that  $a_{n+1} \geq 1$  and  $0 \leq y_n < 1$ , we deduce that

$$\frac{1 + y_n}{1 + z_n} \geq 1 \text{ and } 1 + \frac{1}{a_{n+1}} + z_n \leq 3.$$

This implies that the second factor on the last term of (3.4) is not less than  $1/3$ . Thus, we complete the proof.  $\square$

We are ready to give the proof of the part of liminf in Theorem 2.1.

*Proof of the liminf part.* For any  $x \in [0, 1)$ , by the recursive formula (1.2), we know that

$$q_n(x) = a_n(x)q_{n-1}(x) + q_{n-2}(x) \geq a_n(x)q_{n-1}(x) \geq \dots \geq a_n(x) \cdots a_1(x).$$

Hence that

$$\left\{ x \in \mathbb{I} : \frac{\log q_n(x)}{n} \geq b + \varepsilon \right\} \supseteq \left\{ x \in \mathbb{I} : \frac{\log a_1(x) \cdots a_n(x)}{n} \geq b + \varepsilon \right\}. \quad (3.5)$$

Let  $b_\varepsilon$  be the smallest integer no less than  $e^{b+\varepsilon}$ . Since

$$\left\{ x \in \mathbb{I} : \frac{\log a_1(x) \cdots a_n(x)}{n} \geq b + \varepsilon \right\} \supseteq \left\{ x \in \mathbb{I} : \log a_1(x) \geq b + \varepsilon, \dots, \log a_n(x) \geq b + \varepsilon \right\}$$

and

$$\left\{ x \in \mathbb{I} : \log a_1(x) \geq b + \varepsilon, \dots, \log a_n(x) \geq b + \varepsilon \right\} \supseteq \left\{ x \in \mathbb{I} : a_1(x) = b_\varepsilon, \dots, a_n(x) = b_\varepsilon \right\},$$

combing these with (3.5), we deduce that

$$\begin{aligned} \lambda \left\{ x \in \mathbb{I} : \frac{\log q_n(x)}{n} \geq b + \varepsilon \right\} &\geq \lambda \{ x \in \mathbb{I} : a_1(x) = b_\varepsilon, \dots, a_n(x) = b_\varepsilon \} \\ &= |I(\underbrace{b_\varepsilon, \dots, b_\varepsilon}_n)| \\ &\geq \frac{1}{3b_\varepsilon^2} \cdot |I(\underbrace{b_\varepsilon, \dots, b_\varepsilon}_{n-1})|, \end{aligned}$$

where the last inequality follows from Lemma 3.2. Repeating this procedure  $(n-1)$  times, we obtain that

$$\begin{aligned} \lambda \left\{ x \in \mathbb{I} : \frac{\log q_n(x)}{n} \geq b + \varepsilon \right\} &\geq \left( \frac{1}{3b_\varepsilon^2} \right)^{n-1} \cdot |I(b_\varepsilon)| \\ &= \left( \frac{1}{3b_\varepsilon^2} \right)^{n-1} \cdot \frac{1}{b_\varepsilon(b_\varepsilon + 1)} \geq \left( \frac{1}{3b_\varepsilon^2} \right)^n. \end{aligned}$$

Therefore,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in \mathbb{I} : \frac{\log q_n(x)}{n} \geq b + \varepsilon \right\} \geq -2 \log b_\varepsilon - \log 3.$$

This gives a lower bound of the desired result. Next we will point out that  $-2 \log b_\varepsilon - \log 3 \leq \theta_1(\varepsilon)$ . By the definition of  $b_\varepsilon$ , we know that  $-2 \log b_\varepsilon \leq -2(b + \varepsilon)$ . It follows from Remark 2.2 that  $-(b + \varepsilon) \leq \theta_1(\varepsilon)$ . So  $-2 \log b_\varepsilon - \log 3 < \theta_1(\varepsilon)$ .  $\square$

**3.2. Proof of Theorem 2.3.** The proof of Theorem 2.3 is similar to the proof Theorem 2.1.

*Completion of the proof of Theorem 2.3.* We first prove the limsup part. Let  $t > 0$  be a parameter. Being similar to the proofs of the inequalities (3.2)–(3.3), we deduce that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in \mathbb{I} : \frac{\log q_n(x)}{n} \leq b - \varepsilon \right\} \leq t(b - \varepsilon) + P(1 + t/2).$$

for any  $t > 0$ . Therefore,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in \mathbb{I} : \frac{\log q_n(x)}{n} \leq b - \varepsilon \right\} \leq \theta_2(\varepsilon)$$

with

$$\theta_2(\varepsilon) = \inf_{t > 0} \{ t(b - \varepsilon) + P(1 + t/2) \}.$$

Now we show that  $\theta_2(\varepsilon) < 0$ . For any  $u > -1$ , we consider the function

$$h(u) = u(b - \varepsilon) + P(1 + u/2).$$

Notice that  $h(0) = 0$  and  $h'(0) < 0$  by (1.4), then  $h(t) < 0$  for  $t$  sufficiently close to 0 and hence that  $\theta_2(\varepsilon) < 0$ .

Next we give the proof of the liminf part. It follows from the recursive formula (1.2) that

$$q_n(x) = a_n(x)q_{n-1}(x) + q_{n-2}(x) \leq (a_n(x) + 1)q_{n-1}(x) \leq \dots \leq (a_n(x) + 1) \dots (a_1(x) + 1).$$

So,

$$\left\{x \in \mathbb{I} : \frac{\log q_n(x)}{n} \leq b - \varepsilon\right\} \supseteq \left\{x \in \mathbb{I} : \frac{\sum_{k=1}^n \log(a_k(x) + 1)}{n} \leq b - \varepsilon\right\}. \quad (3.6)$$

Let  $b_\varepsilon^*$  be the largest integer less than or equal to  $(e^{b-\varepsilon} - 1)$ . Here we remark that the assumption  $0 < \varepsilon \leq b - \log 2$  in Theorem 2.3 is just to guarantee that  $b_\varepsilon^* \geq 1$ . Notice that the right-hand set in (3.6) contains the following set

$$\left\{x \in \mathbb{I} : \log(a_1(x) + 1) \leq b - \varepsilon, \dots, \log(a_n(x) + 1) \leq b - \varepsilon\right\}$$

and this subset also contains the following cylinder

$$\left\{x \in \mathbb{I} : a_1(x) = b_\varepsilon^*, \dots, a_n(x) = b_\varepsilon^*\right\},$$

combing this with (3.6), we obtain that

$$\lambda \left\{x \in \mathbb{I} : \frac{\log q_n(x)}{n} \leq b - \varepsilon\right\} \geq |I(\underbrace{b_\varepsilon^*, \dots, b_\varepsilon^*}_n)| \geq \left(\frac{1}{3b_\varepsilon^{*2}}\right)^n,$$

where the last inequality is from Lemma 3.2. Therefore,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{x \in \mathbb{I} : \frac{\log q_n(x)}{n} \leq b - \varepsilon\right\} \geq -2 \log b_\varepsilon^* - \log 3.$$

□

### 3.3. Proof of Theorem 2.5.

*Completion of the proof of Theorem 2.5.* For any  $\varepsilon > 0$  and  $n \geq 1$ , since

$$\begin{aligned} & \lambda \left\{x \in \mathbb{I} : \left| \frac{\log q_n(x)}{n} - b \right| \geq \varepsilon\right\} \\ &= \lambda \left\{x \in \mathbb{I} : \frac{\log q_n(x)}{n} \geq b + \varepsilon\right\} + \lambda \left\{x \in \mathbb{I} : \frac{\log q_n(x)}{n} \leq b - \varepsilon\right\}, \end{aligned} \quad (3.7)$$

we obtain that

$$\begin{aligned} & \limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{x \in \mathbb{I} : \left| \frac{\log q_n(x)}{n} - b \right| \geq \varepsilon\right\} \\ &= \limsup_{n \rightarrow \infty} \frac{1}{n} \log \left( \lambda \left\{x \in \mathbb{I} : \frac{\log q_n(x)}{n} \geq b + \varepsilon\right\} + \lambda \left\{x \in \mathbb{I} : \frac{\log q_n(x)}{n} \leq b - \varepsilon\right\} \right) \\ &\leq \max\{\theta_1(\varepsilon), \theta_2(\varepsilon)\} \end{aligned}$$

where the last inequality follows from the limsups in Theorems 2.1 and 2.3. Therefore, for any  $\varepsilon > 0$ , there exist positive real  $\alpha$  (only depending on  $\varepsilon$ ) and positive integer  $N := N_\varepsilon$  such that for all  $n > N$ , we have

$$\lambda \left\{x \in \mathbb{I} : \left| \frac{\log q_n(x)}{n} - b \right| \geq \varepsilon\right\} \leq e^{-\alpha n}. \quad (3.8)$$

For any  $1 \leq n \leq N$ , since the probabilities of the left-hand side in (3.8) are bounded, we can choose sufficiently large  $A$  (only depending on  $\varepsilon$ ) such that

$$\lambda \left\{x \in \mathbb{I} : \left| \frac{\log q_n(x)}{n} - b \right| \geq \varepsilon\right\} \leq A e^{-\alpha n}$$



holds for all  $n \geq 1$ . Thus, the upper bound of the result in Theorem 2.5 is established. The lower bound of the result in Theorem 2.5 can also be obtained using the similar methods.  $\square$

#### 4. APPLICATIONS

In this section, we will apply our results to the following quantities related to the denominator of convergent  $q_n$  in continued fractions. The following notations  $\theta_1$ ,  $\theta_2$ ,  $b_\varepsilon$  and  $b_\varepsilon^*$  are as defined in the Theorems 2.1 and 2.3.

**4.1. Lyapunov exponents.** Lyapunov exponents measure the exponential rate of divergence of infinitesimally close orbits of a dynamical system. These exponents are intimately related with the global stochastic behavior of the system and are fundamental invariants of a dynamical system. Here we define the *Lyapunov exponent* of the continued fraction transformation  $T$  by

$$l(x) := \lim_{n \rightarrow \infty} \frac{1}{n} \log |(T^n)'(x)|$$

if the limit exists. It is well known (see [18]) that there exists a positive constant  $K$  such that for any  $x \in [0, 1)$ ,

$$\frac{1}{2K} q_n^2(x) \leq |(T^n)'(x)| \leq K q_n^2(x).$$

By Theorem 1.1 and this result, we have that  $l(x)$  is constantly  $\pi^2/(6 \log 2)$  for  $\lambda$ -almost all  $x \in [0, 1)$ . Combing this with Theorems 2.1, 2.3 and 2.5, we know

**Theorem 4.1.** *For any  $\varepsilon > 0$ ,*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in [0, 1) : \frac{1}{n} \log |(T^n)'(x)| - 2b \geq \varepsilon \right\} \leq \theta_1(\varepsilon/2)$$

and

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in [0, 1) : \frac{1}{n} \log |(T^n)'(x)| - 2b \geq \varepsilon \right\} \geq -2 \log b_{\varepsilon/2} - \log 3.$$

**Theorem 4.2.** *For any  $0 < \varepsilon \leq 2b$ ,*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in [0, 1) : \frac{1}{n} \log |(T^n)'(x)| - 2b \leq -\varepsilon \right\} \leq \theta_2(\varepsilon/2)$$

and for any  $0 < \varepsilon \leq 2(b - \log 2)$ ,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in [0, 1) : \frac{1}{n} \log |(T^n)'(x)| - 2b \leq -\varepsilon \right\} \geq -2 \log b_{\varepsilon/2}^* - \log 3.$$

**Theorem 4.3.** *For any  $\varepsilon > 0$ , there exist the constants  $A_1, B_1 > 0$  and  $\alpha_1, \beta_1 > 0$  (both only depending on  $\varepsilon$ ) such that for all  $n \geq 1$ , we have*

$$B_1 e^{-\beta_1 n} \leq \lambda \left\{ x \in [0, 1) : \left| \frac{1}{n} \log |(T^n)'(x)| - \frac{\pi^2}{6 \log 2} \right| \geq \varepsilon \right\} \leq A_1 e^{-\alpha_1 n}.$$

**4.2. The growth rate of Diophantine approximation.** For any  $x \in [0, 1)$  with its continued fraction expansion (1.1), we define

$$d(x) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \left| x - \frac{p_n(x)}{q_n(x)} \right|$$

if the limit exists. It is clear to see that this function stands for the rate of rational numbers approximating to real numbers. By Theorem 1.1 and Diophantine inequalities (1.3), we know that the quantity  $d(x) = -\pi^2/(6 \log 2)$  for  $\lambda$ -almost all  $x \in [0, 1)$ . In view of (1.3), we obtain that

**Theorem 4.4.** *For any  $0 < \varepsilon \leq 2b$ ,*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in [0, 1) : \frac{1}{n} \log \left| x - \frac{p_n(x)}{q_n(x)} \right| + 2b \geq \varepsilon \right\} \leq \theta_2(\varepsilon/2)$$

*and for any  $0 < \varepsilon \leq 2(b - \log 2)$ ,*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in [0, 1) : \frac{1}{n} \log \left| x - \frac{p_n(x)}{q_n(x)} \right| + 2b \geq \varepsilon \right\} \geq -2 \log b_{\varepsilon/2}^* - \log 3.$$

**Theorem 4.5.** *For any  $\varepsilon > 0$ ,*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in [0, 1) : \frac{1}{n} \log \left| x - \frac{p_n(x)}{q_n(x)} \right| + 2b \leq -\varepsilon \right\} \leq \theta_1(\varepsilon/2)$$

*and*

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in [0, 1) : \frac{1}{n} \log \left| x - \frac{p_n(x)}{q_n(x)} \right| + 2b \leq -\varepsilon \right\} \geq -2 \log b_{\varepsilon/2} - \log 3.$$

**Theorem 4.6.** *For any  $\varepsilon > 0$ , there exist the constants  $A_2, B_2 > 0$  and  $\alpha_2, \beta_2 > 0$  (both only depending on  $\varepsilon$ ) such that for all  $n \geq 1$ , we have*

$$B_2 e^{-\beta_2 n} \leq \lambda \left\{ x \in [0, 1) : \left| \frac{1}{n} \log \left| x - \frac{p_n(x)}{q_n(x)} \right| + \frac{\pi^2}{6 \log 2} \right| \geq \varepsilon \right\} \leq A_2 e^{-\alpha_2 n}.$$

**4.3. The growth rate of the length of cylinders.** In dynamical system, the theorem of Shannon-McMillan-Breiman (see [4, Theorem 6.2.1]) states that for every generating partition on an ergodic system of finite entropy, the exponential decay rate of the measure of cylinder sets equals the metric entropy almost everywhere. Now we consider the continued fractions dynamical system  $([0, 1), \mathcal{B}, T, \mu)$ , where  $\mathcal{B}$  is the Borel  $\sigma$ -algebra on  $[0, 1)$  and  $\mu$  is the *Gauss measure* with a bounded density  $\frac{1}{(1+x) \log 2}$  on  $[0, 1)$  with respect to Lebesgue measure. For any  $x \in [0, 1)$ , we put

$$s(x) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \mu(I_n(x))$$

if the limit exists, where  $I_n(x)$  denotes the  $n$ -th order cylinder containing  $x$ . It is clear to see that  $s(x)$  is alternatively defined by  $s(x) = \lim_{n \rightarrow \infty} (\log |I_n(x)|)/n$  because of the relation between Gauss measure and Lebesgue measure. Shannon-McMillan-Breiman's theorem yields that  $s(x)$  exists and is equal to  $-\pi^2/(6 \log 2)$  for  $\lambda$ -almost all  $x \in [0, 1)$ . Notice that

$$\frac{1}{2q_n^2(x)} \leq |I_n(x)| = \frac{1}{q_n(x)(q_n(x) + q_{n-1}(x))} \leq \frac{1}{q_n^2(x)}$$

(see [4, 11]), in view of Theorems 2.1, 2.3 and 2.5, we have

**Theorem 4.7.** For any  $0 < \varepsilon \leq 2b$ ,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in [0, 1) : \frac{1}{n} \log |I_n(x)| + 2b \geq \varepsilon \right\} \leq \theta_2(\varepsilon/2)$$

and for any  $0 < \varepsilon \leq 2(b - \log 2)$ ,

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in [0, 1) : \frac{1}{n} \log |I_n(x)| + 2b \geq \varepsilon \right\} \geq -2 \log b_{\varepsilon/2}^* - \log 3.$$

**Theorem 4.8.** For any  $\varepsilon > 0$ ,

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in [0, 1) : \frac{1}{n} \log |I_n(x)| + 2b \leq -\varepsilon \right\} \leq \theta_1(\varepsilon/2)$$

and

$$\liminf_{n \rightarrow \infty} \frac{1}{n} \log \lambda \left\{ x \in [0, 1) : \frac{1}{n} \log |I_n(x)| + 2b \leq -\varepsilon \right\} \geq -2 \log b_{\varepsilon/2} - \log 3.$$

**Theorem 4.9.** For any  $\varepsilon > 0$ , there exist the constants  $A_3, B_3 > 0$  and  $\alpha_3, \beta_3 > 0$  (both only depending on  $\varepsilon$ ) such that for all  $n \geq 1$ , we have

$$B_3 e^{-\beta_3 n} \leq \lambda \left\{ x \in [0, 1) : \left| \frac{1}{n} \log |I_n(x)| + \frac{\pi^2}{6 \log 2} \right| \geq \varepsilon \right\} \leq A_3 e^{-\alpha_3 n}.$$

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